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THESIS

NEAR-OPTIMAL FINITE SOLUTIONS TO THE
THREE AND FOUR STEP DISCRETE EVASION GAMES

by

Scott W. Goodson

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Thesis Advisor: James N. Eagle

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Near-optimal Finite Solutions to the Three and Four Step Discrete Evasion Games

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Submitted in partial fulfillment of the requirements for the degree of

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ABSTRACT

A review of discrete pursuer-evader games and known solutions is presented. A method is given for obtaining a finite memory, near-optimal evader strategy for the three-step game, which greatly reduces data storage requirements from previous near-optimal strategies. Additionally near-optimal evader strategies for the four-step game are discussed.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

I. INTRODUCTION

The discrete time step pursuer-evader game was first described by Rufus Isaacs of the Rand Corporation in the early 1950's in an attempt to look at the problem of attacking a moving target who is maneuvering so as to confound the prediction of his future position. The general problem, as described by Isaacs is as follows:

> A battleship in midocean is aware of an enemy bomber's presence, but the plane is too high for precise detection. The ship is interested only in not being hit; it has no offensive means. The plane has one bomb and we suppose--to avoid extraneous factors--that the bomber's aim is excellent. The battleship knows this, but knows nothing about when or where the bomb will be dropped until after detonation. It is to maneuver so as to minimize the hit probability. . . There is a time lag T between the bomber's last sighting of the ship and detonation. Thus the bomber must aim at an anticipated position of the ship . . . As simple as this problem sounds circumstantially, it is difficult technically. To gain a foothold, we simplified it further. We made the ocean one-dimensional and discrete. That is, we supposed the battleship to be located on one of a long row of points and at each unit of time he hops to one adjoining one, enjoying the sole choice of a right or left jump. The time lag was to be an integral number n of time units, or--the same thing--of jumps. This is tantamount to saying that the bomber knows all positions of the battleship which precede his present one by n jumps or more Ref.[1].

The solution to the single time step game, (i.e. n=1) is trivial but the complexity increases greatly as the time lag or number of time steps increases. Isaacs, upon formulating the game, proposed pursuer and evader strategies to the two-step game, however the proof of the optimality of these

strategies is highly complex. The complexity of the multiple step games arises from the fact that the evader doesn't know when the pursuer will attack; if he did it would be an easy matter for the evader to distribute himself uniformly over the n+1 possible positions at the time of detonation, and limit the pursuer to a kill probability of 1/(n+1). Without knowing the time of attack the evader must attempt to make his position uniform at every time step and this is not possible.

The three-step pursuer-evader game is yet unsolved, however near-optimal strategies for both the pursuer and evader have been described. The best existing evader strategy, developed by Joseph Bram Ref.[2], involves the evader maintaining an infinite memory of probabilities corresponding to the probability of turning given the evader has not turned for the last k moves. This thesis will investigate alternative finite evader strategies to attempt to lower the existing upper bound on the three-step game value while drastically reducing memory requirements and additionally look briefly at possible evader strategies in the four-step game.

II. KNOWN SOLUTIONS AND STRATEGIES FOR PURSUER-EVADER GAMES

A. STRUCTURE

For uniformity, the convention and structure described below will be used hereafter in the description of all discrete n-step pursuer-evader games. The pursuer is the maximizing player who by selection of time of fire and aim point tries to maximize the probability of killing the evader (a kill is achieved when the pursuer fires at the position the evader subsequently occupies n time steps later). The evader is the minimizing player, who by selection of maneuvers along the discrete linear state space, attempts to minimize the probability of being killed. The evader's maneuvers can be described as a sequence of lefts and rights (L and R) with each n-bit sequence of L's and R's corresponding to one of the n+1 final positions achievable in n steps from an arbitrary starting position as shown in Figure 2.1. The above-described mapping of n-bit left-right sequences to final position is symmetric under interchange of L's and R's (i.e. LLR corresponds to a symmetric position to RRL in the three-step case). Due to this symmetry it is equivalent to describe the evader's maneuvers as a sequence of straights and turns (S and T which provides an equivalent mapping in Figure 2.2. A turn signifies the evader moves in the opposite direction to his previous move

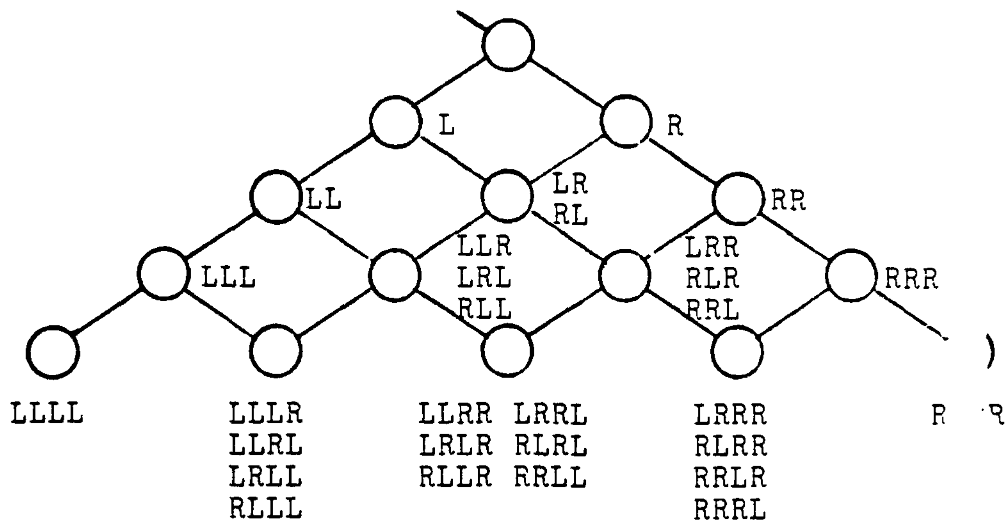

Figure 2.1 Possible Evader Positions in n Steps.

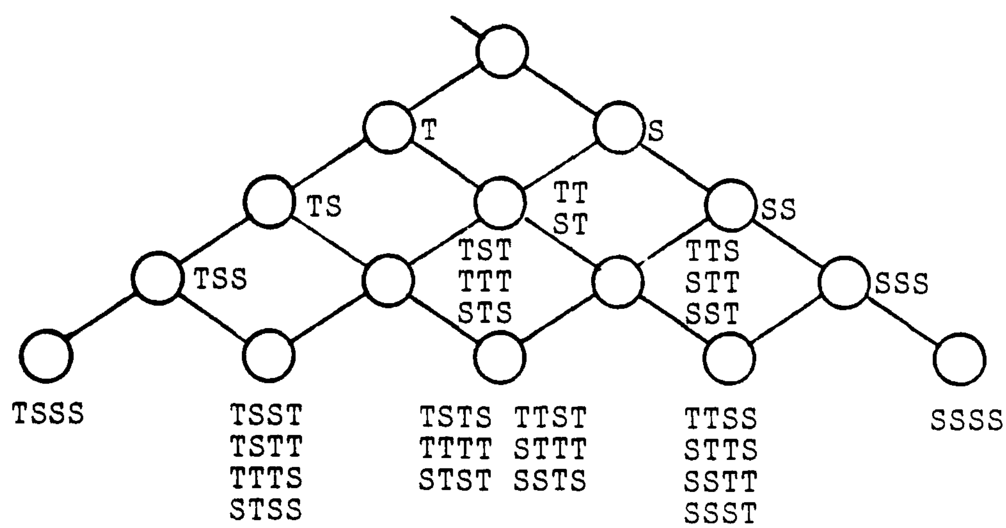

Figure 2.2 Possible Evader Positions in terms of Straights and Turns.

and a straight signifies he continues in the same direction as his previous move. Any n-bit sequence of lefts and rights can be translated into an equivalent (n-1) bit sequence of straights and turns (i.e. LRRL becomes TST). Note that in general there may be several possible sequences of turns and straights which lead to the same final position (for n=3, TST, TTT, and STS all result in the evader occupying the position one step to the left of his original position).

B. ONE-STEP GAME

The single step pursuer-evader game has a simple solution. With only one time step elapsing between the pursuer's time of fire and weapon detonation the evader can always distribute himself uniformly over the two positions achievable in one step shown in Figure 2.3. The evader on each step can continue straight with probability (1-p) or turn with probability p. Since the intelligent pursuer will limit his shot to one of the two feasible positions of the evader when he fires (position 1 or 2 of Figure 2.3), the game can be represented graphically as shown in Figure 2.4. The minimax solution occurs when p=0.5. The corresponding value of the game is 0.5. The optimal evader strategy is to fire at position 1 or 2 with equal probability.

C. TWO-STEP GAME

The two-step pursuer-evader game is not nearly as simple in its solution as the one-step game. The solution was

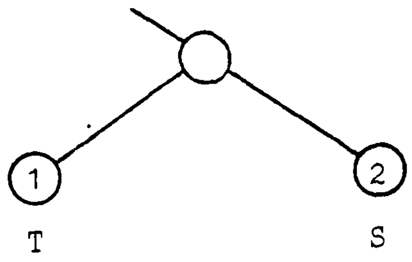

Figure 2.3 Achievable Evader Positions in One-Step Game

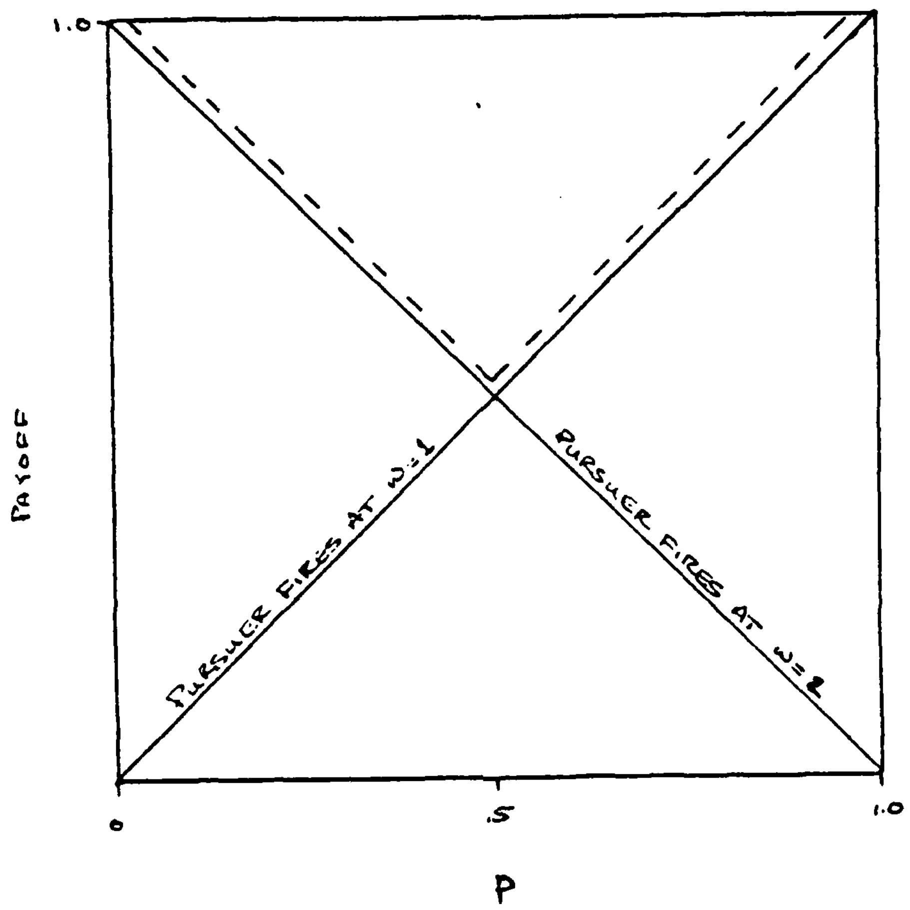

Figure 2.4 Graphical Solution to the One-Step Game.

found by starting with the hypothesis that the evader's maneuver will depend only on his previous maneuver and none earlier; thus the probability of continuing in the same direction as the last move is denoted by $(1-p)$, with p being the probability of moving in the opposite direction to the previous move. The attainable positions for the evader and the corresponding probabilities under the above hypothesis are shown in Figure 2.5. The pursuer can be expected to select the position (1, 2 or 3) with the highest associated probability. The evader will select p so as to minimize this maximum probability. The optimal value of p is then found by solving:

$$\min_{p} \left[\text{MAX} \quad \{p-p^2,\ p,\ (1-p)^2\} \right]$$

$$\text{s.t. } 0 \leq p \leq 1.0$$

Graphically the solution is shown in Figure 2.6. The resulting solution is found by solving the quadratic $p=(1-p)^2$ which has a root at $p=(3-\sqrt{5})/2 = 0.38197\ldots$; this value is also the probability that the evader is in position 2 or 3 of Figure 2.5 and thus the value of the game. The proof that this evader strategy is optimal and that $(3-\sqrt{5})/2$ is the value of the game is complex. Three different proofs are given by Dubins Ref.[3], Isaacs Ref.[4] and Ferguson Ref.[5]. The pursuer strategies in the multi-step games are characterized by the non-existence of an optimal strategy; the pursuer can always increase his expected

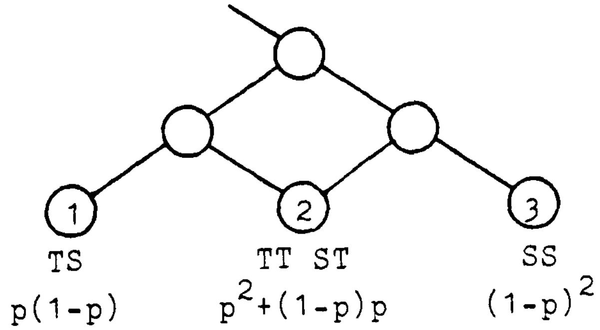

Figure 2.5 Achievable Evader Positions in Two-Step Game.

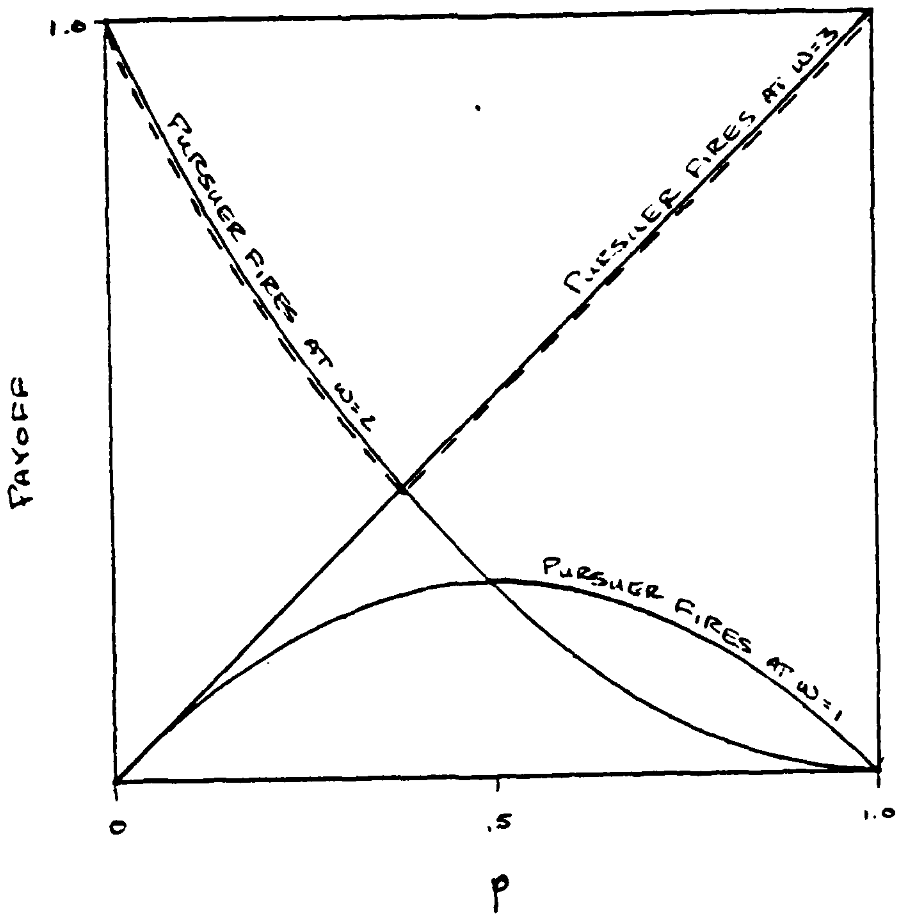

Figure 2.6 Graphical Solution to the Two-Step Game.

kill probability by waiting a few more time periods but he cannot wait indefinitely to fire or his payoff is zero. This contradiction leads to strategies for the pursuer which have payoffs arbitrarily close to, but not equal to, the value of the game. Ferguson developed such a pursuer strategy which confirmed that $(3-\sqrt{5})/2 = 0.38197 \ldots$ was the value of the two-step game.

D. THREE-STEP GAME

As stated earlier the three-step pursuer-evader game is yet unsolved. The value of the three-step game has been bounded to:

$$0.28423 \leq v \leq 0.28903$$

by Bram. This section will investigate previous near-optimal evader strategies for the three-step game and the resulting upper bounds upon the game value.

1. Markov Hypothesis Strategy

The Markov Hypothesis for the n-step pursuer-evader game is stated as follows: the probability that the evader will go left or right (or, straight or turn) is dependent on the previous n-1 moves but not on any moves further in the past than the n-1st. This form or evader strategy makes intuitive sense since it does not seem likely that an optimal evader strategy will depend upon information which the pursuer already knows at the time of fire. The known optimal strategies for the one and two-step games adhere to

the Markov Hypothesis. In the one-step game the optimal evader turns or continues straight with equal probability, therefore independent of all previous moves. (i.e. $P(S) = P(T) = P(L) = P(R)$). In the two-step game the optimal evader uses a strategy where the probability of turning (or continuing straight) depends only upon his previous move (i.e. $P(S) = P(L|L) = P(R|R) = 0.61803$ and $P(T) = P(L|R) = P(R|L) = 0.38197$).

The Markov Hypothesis will now be applied to the three-step game. Since the evader will condition his next move upon his previous two moves, his strategy can be described by a 2x2 transition matrix as shown in Figure 2.7. The state of the evader at any time is S or T since this state is a function of the evader's last two moves (i.e. LL or RR→S). In this transition matrix:

$$q_1 = P(\text{Next state is S} \mid \text{Last state was S})$$
$$q_2 = P(\text{Next state is S} \mid \text{Last state was T}).$$

The four achievable positions for the evader in the three-step game and the associated maneuver sequences are shown in Figure 2.8. Let the variable W represent the final position of the evader three steps after the time of fire; from Figure 2.8 it can be seen $W \varepsilon (1,2,3,4)$. Let the variable STATE represent the state (S or T) that the evader occupies at the time of fire. The probability that the evader occupies any final position is a function of q_1 and q_2 when

$$
\begin{array}{ccc}
 & & \text{NEXT MOVE} \\
 & & \begin{array}{cc} \text{S} & \qquad\qquad \text{T} \end{array} \\
\begin{array}{c} \text{LAST} \\ \text{MOVE} \end{array} & \begin{array}{c} \text{S} \\ \text{T} \end{array} & \begin{bmatrix} q_1 & 1-q_1 \\ q_2 & 1-q_2 \end{bmatrix}
\end{array}
$$

Figure 2.7 Markov Hypothesis Transition Matrix for Three-Step Game.

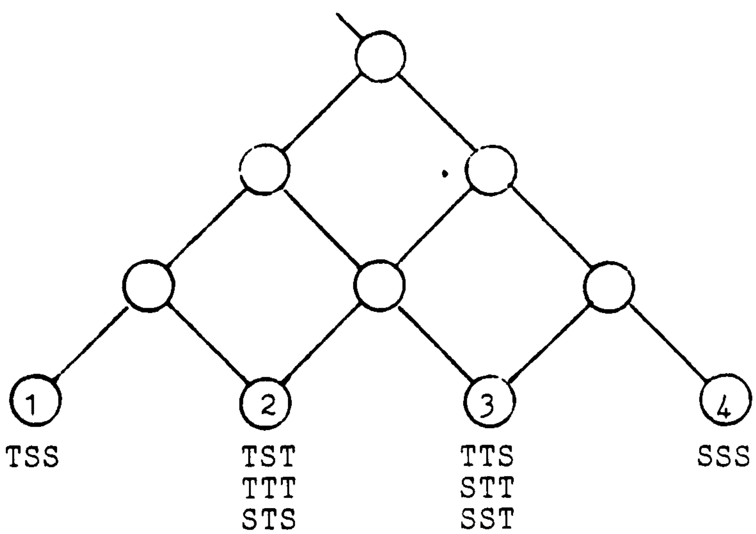

Figure 2.8 Achievable Evader Positions in Three-Step Game.

conditioned upon his initial state. For example, given STATE=S, to arrive at W=1, the sequence of transitions undergone must be:

S to T to S to S

The probability of this occurrence can be written:

$$P(W=1|STATE=S)=(1-q_1)q_2q_1$$

The remaining seven conditional probabilities are:

$$P(W=2|STATE=S)=(1-q_1)q_2(1-q_1)+(1-q_1)(1-q_2)^2+q_1(1-q_1)q_2$$
$$P(W=3|STATE=S)=(1-q_1)(1-q_2)q_2+q_1(1-q_1)(1-q_2)+q_1^2(1-q_1)$$
$$P(W=4|STATE=S)=q_1^3$$
$$P(W=1|STATE=T)=(1-q_2)q_2q_1$$
$$P(W=2|STATE=T)=(1-q_2)q_2(1-q_1)+(1-q_2)^3+q_2(1-q_1)q_2.$$
$$P(W=3|STATE=T)=(1-q_2)^2q_2+q_2(1-q_1)(1-q_2)+q_2q_1(1-q_1)$$
$$P(W=4|STATE=T)=q_2q_1^2$$

At any time the pursuer may choose to fire, he knows which of the two states (S or T) that the evader is in by observing his last two moves. The optimal values of q_1 and q_2 under this strategy are found by solving the following non-linear problem:

$$\min_{q_1q_2} \left[\max_{i,j} \{P(w=j|STATE=i)\} \right] \quad \begin{matrix} j=1,2,3,4 \\ i=S,T \end{matrix}$$

The solution, due to Ferguson, is $q_1 = 0.63397. . .$, $q_2 = 0.73205. . .$ with a corresponding game value of 0.29423, the resulting matrix of conditional probabilities is shown in Table I. Ferguson states when presenting this evader strategy, that it is not known to be optimal and in fact he conjectures that no evader strategy of finite dependence is optimal for the evader. The strategy of Bram presented in the next section will show that indeed an evader strategy of infinite dependence does result in a tighter bound on the game value.

2. Infinite Dependence Strategy

As mentioned in Chapter I, the best existing evader strategy for the three-step game was described by Joseph Bram. This strategy can be described as an infinite sequence of the conditional probabilities that the evader will continue straight given the state S of his previous moves. If the previous move by the evader was a turn, the evader is in state S=1, while if the previous k-1 moves have been straight the evader is in state S=k. (Note that the state space of S is infinite). We will denote a turn by T and a straight by S as before. At each time step the evader continues straight or turns with a probability dependent upon his state S. Let:

$$p_k = P(\text{Straight}|S=k).$$

If the evader is in state k at some time n, at time n+3 the evader can be in one of four positions described by W

TABLE I

P($W=\hat{W}$|STATE) for Three-Step Markov Hypothesis Strategy

$$q_1 = P(S|S) = 0.63397$$

$$q_2 = P(S|T) = 0.73205$$

$\hat{W}=$	1	2	3	4
STATE				
S	.16987	.29423	.28109	.25480
T	.12435	.28719	.29423	.29423

previously. There are eight possible 3-bit sequences of S's and T's which correspond to the four possible terminal positions as shown in Figure 2.8. The probabilities associated with each position W given k are as follows:

$$P(W=1|S=k)=(1-p_k)p_1p_2$$
$$P(W=2|S=k)=(1-p_k)p_1(1-p_2)+(1-p_k)(1-p_1)^2+p_k(1-p_{k+1})p_1$$
$$P(W=3|S=k)=(1-p_k)(1-p_1)p_1+p_k(1-p_{k+1})(1-p_1)+p_kp_{k+1}(1-p_{k+2})$$
$$P(W=4|S=k)=p_kp_{k+1}p_{k+2}$$

If the evader fires at time n, at position W, when S=k, his expected payoff will be:

$$P(W=\hat{W}|S=k)$$

The upper bound on the value of the game played with this strategy is:

$$\max_{k} \max_{\hat{W}} \{P(W=W|S=k)\}$$

The evader of course will attempt to select his infinite array of P_k's so as to minimize the above bound which is the maximum payoff that the pursuer can achieve. The best set of P_k's found by Bram is delineated in Table II, while the resulting $P(W=\hat{W}|S=k)$ is shown in Table III. The upper bound on the game value under this specific set of P_k's is the maximum value found in Table III or 0.28903. In this strategy the decision to turn or continue straight has a

TABLE II

A Safe Set of p_k's for the Evader

k	p_k
1	.69290
2	.62467
3	.66775
4	.65137
5	.66241
6	.65859
7	.66135
8	.66047
9	.66116
10	.66096
11	.66114
12	.66109
13	.66114
14	.66113
15	.66114
.	.
.	.
.	.

TABLE III

$P(W=\hat{W}|S=k)$ using p_k's of Table II

k \ $\hat{W}=$	1	2	3	4
1	.13292	.28903	.28903	.28903
2	.16246	.27682	.28903	.27170
3	.14381	.27905	.28903	.28818
4	.15090	.27591	.28903	.28417
5	.14612	.27634	.28903	.28852
6	.14778	.27552	.28903	.28768
7	.14658	.27560	.28903	.28880
8	.14696	.27539	.28903	.28863
9	.14666	.27539	.28903	.28892
10	.14675	.27534	.28903	.28889
11	.14667	.27534	.28903	.28896
12	.14669	.27532	.28903	.28896
13	.14667	.27532	.28903	.28898
.	.	.	.	.

dependence upon the previous moves. That dependence may extend infinitely far back; thus the evader is required to maintain the infinite array of P_k's to execute this near-optimal strategy.

3. Sub-Markov Strategy

The strategy presented here is due to Bouchoux Ref.[6] and is characterized by a strategy where the evader's sequence of moves is not Markovian in itself but one in which that sequence is generated by a substructure which is Markovian, hence the description Sub-Markov. This form of strategy is suggested by its use in providing optimal strategies in emission-prediction games described by Blackwell Ref.[7] and Matula Ref.[8]. The pursuer-evader game, while similar to emission-prediction games, is complicated by the fact that there are several distinct sequences of moves which lead to the possible terminal positions. Since the pursuer (predictor) must fire at one of those terminal points and not at a specific sequence of moves, the game is more complex. Bouchoux describes a strategy based upon three states, A, B and C, through which the evader transitions in a Markovian manner. When in state A the evader always turns, while in states B and C he always goes straight. After each move, straight or turn, the evader transitions between states according to a 3x3 transition matrix and is ready for his next move. This strategy is finite in the memory required by the evader and Bouchoux

obtained a bound on the game value of 0.28922 by optimizing upon the transition matrix.

III. EXTENDED MARKOV STRATEGY

A. MOTIVATION AND DESCRIPTION

The evader strategy to be investigated will be called Extended Markov because it is an extension of the finite dependence of the Markov Hypothesis strategy. The dependence will be finite but will extend beyond the previous n-1 steps. In the Markov Hypothesis strategy, for the three-step game, discussed in II.D.1., the best strategy for the evader resulted in an upper bound on the game value of 0.29423. If the dependence is restricted to only the previous move instead of the previous two moves the best strategy results in an upper bound of 0.29630 (Note: this is equivalent to adding the constraint $q_1=q_2$ to the nonlinear problem described in II.D.1. with a solution at $q_1=q_2=2/3$). Since Bram's strategy showed that the Markov Hypothesis was not optimal for the three-step game, it seems that a Markovian strategy where the dependence is finite but extends beyond the last n-1 moves might result in a tighter bound on the game value than previously obtained. This is the class of strategies to be called Extended Markov. These strategies for the three-step game, Markovian in nature, will arise from a dependence upon the last three or more moves and will be called the n-dependent strategies where n represents the level of dependence. In this context, the

Markov Hypothesis strategy for the three-step game is the two-dependent strategy.

B. GENERAL N-DEPENDENT STRATEGY

In the n-dependent strategy the evader will determine his next move based upon his previous n moves. The evader can be thought of as controlling 2^n variables, each being the probability of going (say) right given the previous n steps have been in a certain sequence. We will utilize the left-right symmetry of the problem by considering only paths where the last move is to the (say) right, resulting in only 2^{n-1} variables, each representing the probability of going (say) straight given the last n steps have produced a certain n-1 bit sequence of straights and turns. The general n-dependent strategy can be described by a Markov chain having 2^{n-1} states corresponding to the 2^{n-1} different n-1 bit sequences of straights and turns which are possible based on the last n moves (i.e. conditioning upon the last n moves is equivalent to conditioning on the last n-1 straights or turns). From each of the 2^{n-1} states there is a fixed probability that the evader will maneuver to one of the four final positions W in the next three steps. A 2^{n-1} x 2^{n-1} transition matrix will be used to describe the conditional probability of turning or continuing straight given the current state ((n-1)-bit sequence). Since the state describes the previous n moves in terms of straights and turns only two possible transitions exist from each of the

states. The first n-2 bits of the state transitioned to are determined by the last n-2 bits of the state transitioned from; the last bit will be S or T depending upon the new move. Due to this structure the transition matrix will be completely defined by 2^{n-1} variables (called q_i $i=1, 2^{n-1}$) which represent the probability of continuing straight given the current state. The other transition probability for that state will obviously be $(1-q_i)$. Using a transition matrix so constructed, the conditional probability of ending in one of the four final positions (W=1,2,3 or 4) can be found. In order to arrive in position 1, for example, the sequence of states transitioned must result in the terminating three-bit sequence, TSS, as can be seen from Figure 2.8. Thus $P(W=\hat{W}|STATE)$ is a function of the variables q_i $(i=1, 2^{n-1})$ and the best n-dependent strategy is solved by the following non-linear program:

$$\min_{q_i} \left[\max_{\hat{W},STATE} \quad P(W=\hat{W}|STATE) \right]$$

$$\text{s.t.} \quad 0 \leq q_i \leq 1.0 \qquad i=1,2$$

For general n, it is seen that the above program involves minimizing the maximum of 2^{n+1} (4 positions x 2^{n-1} states) non-linear functions of up to 2^{n-1} variables. No analytic solution has been found and in later sections near-optimal solutions will be found by non-linear search techniques.

C. THREE-DEPENDENT STRATEGY

The first extension of the Markov Hypothesis strategy is the three-dependent strategy described by four states (SS, ST, TS, TT) and a 4x4 transition matrix shown in Figure 3.1 where:

$$q_1 = P(\text{next move is straight} \mid \text{State is SS})$$

or equivalently;

$$q_1 = P(\text{next state is SS} \mid \text{last state was SS})$$

The sixteen conditional probabilities of terminating in one of the four positions W, given the evader starts from one of the four states are listed in Table IV. The best solution found using the three-dependent strategy gives an upper bound on the game value of 0.28964 when:

$$q_1 = 0.66163 \qquad q_3 = 0.62489$$
$$q_2 = 0.70054 \qquad q_4 = 0.70054$$

The matrix of conditional probabilities evaluated at this point are in Table V. This solution was found by utilizing an improved feasible direction search which was started from a known "good" solution. For the three-dependent strategy a good starting point is found by applying the known two-dependent (Markov Hypothesis) solution to the three-dependent structure. If one applies the restriction $q_1=q_3$ and $q_2=q_4$ to the three-dependent strategy, it is equivalent

$$
\begin{array}{cc}
 & \text{NEXT STATE} \\
 & \begin{array}{ccccc} & SS & ST & TS & TT \end{array} \\
\begin{array}{c} \text{LAST} \\ \text{STATE} \end{array} &
\begin{array}{c} SS \\ ST \\ TS \\ TT \end{array}
\left[\begin{array}{cccc}
q_1 & 1-q_1 & 0 & 0 \\
0 & 0 & q_2 & 1-q_2 \\
q_3 & 1-q_3 & 0 & 0 \\
0 & 0 & q_4 & 1-q_4
\end{array}\right]
\end{array}
$$

Figure 3.1 4x4 Transition Matrix for 3-Dependent Strategy.

TABLE IV

$P(W=\hat{W}|STATE)$ for 3-Dependent Strategy

Notation: $p_i = 1-q_i \quad i=1,2,3,4$

$$P(W=1|SS) = p_1q_2q_3$$
$$P(W=2|SS) = p_1q_2p_3 + p_1p_2p_4 + q_1p_1q_2$$
$$P(W=3|SS) = p_1p_2q_4 + q_1p_1p_2 + q_1q_1p_1$$
$$P(W=4|SS) = q_1q_1q_1$$
$$P(W=1|ST) = p_2q_4q_3$$
$$P(W=2|ST) = p_2q_4p_3 + p_2p_4p_4 + q_2p_3q_2$$
$$P(W=3|ST) = p_2p_4q_4 + q_2p_3p_2 + q_2q_3p_1$$
$$P(W=4|ST) = q_2q_3q_1$$
$$P(W=1|TS) = p_3q_2q_3$$
$$P(W=2|TS) = p_3q_2p_3 + p_3p_2p_4 + q_3p_1q_2$$
$$P(W=3|TS) = p_3p_2q_4 + q_3p_1p_2 + q_3q_1p_1$$
$$P(W=4|TS) = q_3q_1q_1$$
$$P(W=1|TT) = p_4q_4q_3$$
$$P(W=2|TT) = p_4q_4p_3 + p_4p_4p_4 + q_4p_3q_2$$
$$P(W=3|TT) = p_4p_4q_4 + q_4p_3p_2 + q_4q_3p_1$$
$$P(W=4|TT) = q_4q_3q_1$$

TABLE V

Good Evader Strategy in 3-Dependent Case

$$q_1 = P(S|SS) = 0.66163$$
$$q_2 = P(S|ST) = 0.70054$$
$$q_3 = P(S|TS) = 0.62489$$
$$q_4 = P(S|TT) = 0.70054$$

	$P(W=\hat{W}\|STATE)$			
$\hat{W}=$	1	2	3	4
STATE				
SS	.14812	.27609	.28615	.28964
ST	.13109	.28964	.28964	.28964
TS	.16421	.28033	.28191	.27355
TT	.13109	.28964	.28964	.28964

TABLE VI

Good Evader Strategy in 4-Dependent Case

$$q_1 = P(S|SSS) = 0.65931 \qquad q_5 = P(S|TSS) = 0.66543$$
$$q_2 = P(S|SST) = 0.69579 \qquad q_6 = P(S|TST) = 0.69579$$
$$q_3 = P(S|STS) = 0.62474 \qquad q_7 = P(S|TTS) = 0.62474$$
$$q_4 = P(S|STT) = 0.69579 \qquad q_8 = P(S|TTT) = 0.69579$$

	$P(W=\hat{W}\|STATE)$			
$\hat{W}=$	1	2	3	4
STATE				
SSS	.14809	.27677	.28854	.28659
SST	.13224	.28925	.28925	.28925
STS	.16312	.27814	.28465	.27409
STT	.13224	.28925	.28925	.28925
TSS	.14543	.27606	.28925	.28925
TST	.13224	.28925	.28925	.28925
TTS	.16312	.27814	.28465	.27409
TTT	.13224	.28925	.28925	.28925

to the strategy discussed in II.D.1. with an upper bound of 0.29423 when:

$$q_1 = q_3 = 0.63397 \qquad q_2 = q_4 = 0.73205$$

Analogously any near-optimal solution to the n-dependent strategy will provide a "good" initial solution to the (n+1)-dependent strategy. While the solution given above for the three-dependent strategy is not known to be optimal, but rather a local minimum of the problem described in III.B., it does represent a significant improvement over the two-dependent strategy (0.29423) and is close in value to the infinite strategy of Bram (0.28903). Appendix A presents an analysis of the above three-dependent solution and shows that the proposed solution does satisfy first-order Kuhn-Tucker conditions (necessary but not sufficient) for a global minimum. It is interesting to note that in the proposed solution $q_2=q_4$ or:

$$P(S|ST) = P(S|TT).$$

Additionally in order for the pursuer to receive his maximum achievable payoff he must refrain from attacking when the state is TS or be limited to a payoff of 0.28191.

D. FOUR AND FIVE-DEPENDENT STRATEGIES

The treatment of the four-dependent and five-dependent strategies is equivalent to the previously described three-dependent strategy with the expansion of the state space and

number of variables involved to eight and sixteen respectively. Good solutions to the four and five-dependent strategies were found, as in the three-dependent case, by starting at a known near-optimal set of values for the q_i's and conducting an improving feasible direction search until a local minimum was found. The best solutions thus found to the four and five-dependent strategies and the resulting conditional probability matricies are shown in Tables VI and VII.

E. CHARACTERISTICS OF THREE, FOUR AND FIVE-DEPENDENT STRATEGIES

The solutions found for the three, four and five-dependent strategies, outlined in Tables V, VI and VII show several revealing characteristics. In each case the conditional probability of continuing straight given the n-1 bit state is not dependent upon all of the information contained in that n-1 bit sequence. The probabilities are dependent only upon the number of time steps elapsed since the last turn maneuver and not upon any turn-straight information further in the past than that last turn. For example, letting t denote the number of time steps since the last turn, then in the five-dependent solution:

$$q_2=q_4=q_6=q_8=q_{10}=q_{12}=q_{14}=q_{16}= P(S|t=1)$$
$$q_3=q_7=q_{11}=q_{15} = P(S|t=2)$$
$$q_5=q_{13} = P(S|t=3)$$

TABLE VII

Good Evader Strategy in 5-Dependent Case

$q_1 = P(S|SSSS) = 0.66120$
$q_2 = P(S|SSST) = 0.69385$
$q_3 = P(S|SSTS) = 0.62470$
$q_4 = P(S|SSTT) = 0.69385$
$q_5 = P(S|STSS) = 0.66698$
$q_6 = P(S|STST) = 0.69385$
$q_7 = P(S|STTS) = 0.62470$
$q_8 = P(S|STTT) = 0.69385$
$q_9 = P(S|TSSS) = 0.65034$
$q_{10} = P(S|TSST) = 0.69385$
$q_{11} = P(S|TSTS) = 0.62470$
$q_{12} = P(S|TSTT) = 0.69385$
$q_{13} = P(S|TTSS) = 0.66698$
$q_{14} = P(S|TTST) = 0.69385$
$q_{15} = P(S|TTTS) = 0.62470$
$q_{16} = P(S|TTTT) = 0.69385$

$P(W=\hat{W}|\text{STATE})$

$\hat{W}=$ STATE	1	2	3	4
SSSS	.14685	.27541	.28867	.28907
SSST	.13270	.28910	.28910	.28910
SSTS	.16267	.28569	.28910	.27097
SSTT	.13270	.28910	.28910	.28910
STSS	.14435	.27975	.28910	.28680
STST	.13270	.28910	.28910	.28910
STTS	.16267	.28569	.28910	.27097
STTT	.13270	.28910	.28910	.28910
TSSS	.15156	.27670	.28742	.28432
TSST	.13270	.28910	.28910	.28910
TSTS	.16267	.28569	.28910	.27097
TSTT	.13270	.28910	.28910	.28910
TTSS	.14435	.27975	.28910	.28680
TTST	.13270	.28910	.28910	.28910
TTTS	.16267	.28569	.28910	.27097
TTTT	.13270	.28910	.28910	.28910

$q_9 \quad = P(S|t=4)$

$q_1 \quad = P(S|t>4)$

It is hypothesized that this characteristic holds for the optimal form of any n-dependent strategy. If this is so it can be seen that the n-dependent strategy is a finite (truncated) version of the Bram strategy presented in II.D.2. and as the level of dependence n is increased without bound the bound of 0.28903 of Bram is expected to hold.

Each of the investigated strategies is also characterized by having some states in which the evader must refrain from firing, else he forfeits his ability to maximize his payoff. As the level of dependence increases however, the penalty to the pursuer who fires when the evader is in one of these states diminishes. Table III shows that under Bram's strategy there is no time at which the pursuer cannot achieve his maximum payoff given he always fires at position W=3.

IV. FOUR-STEP GAME

The four-step pursuer-evader game has been the subject of little interest due to the unsolved nature of the three-step game. We shall briefly look at the four-step game and discover that the apparent characteristic structure of the three-step extended Markov strategies does not extend to the four-step game. Given a four-step time delay between the attacker's time of fire and subsequent detonation, the evader may achieve five different positions through the sixteen different four-bit sequences of turns and straights as shown in Figure 4.1. The Markov Hypothesis strategy solution to the four-step game is due to Washburn Ref.[9]. In the four-step game the Markov Hypothesis has dependence extending to the last three moves, the best strategy under this hypothesis bounds the value of the game to 0.23740 or below, the q values and resulting conditional probability matrix is shown in Table VIII. The first extended Markov strategy of the four-step game, the only one investigated, is the four-dependent strategy; in this strategy dependence reaches back to the last four moves. The best solution found using the four-dependent strategy is shown in Table IX and provides an upper bound of 0.23734. While this is an improvement over the Markov Hypothesis solution of Washburn, the improvement is very slight. Additionally, no underlying characteristic

such as discussed in III.E. for the three-step extended Markov strategies is apparent from the three and four-dependent strategies investigated for the four-step game.

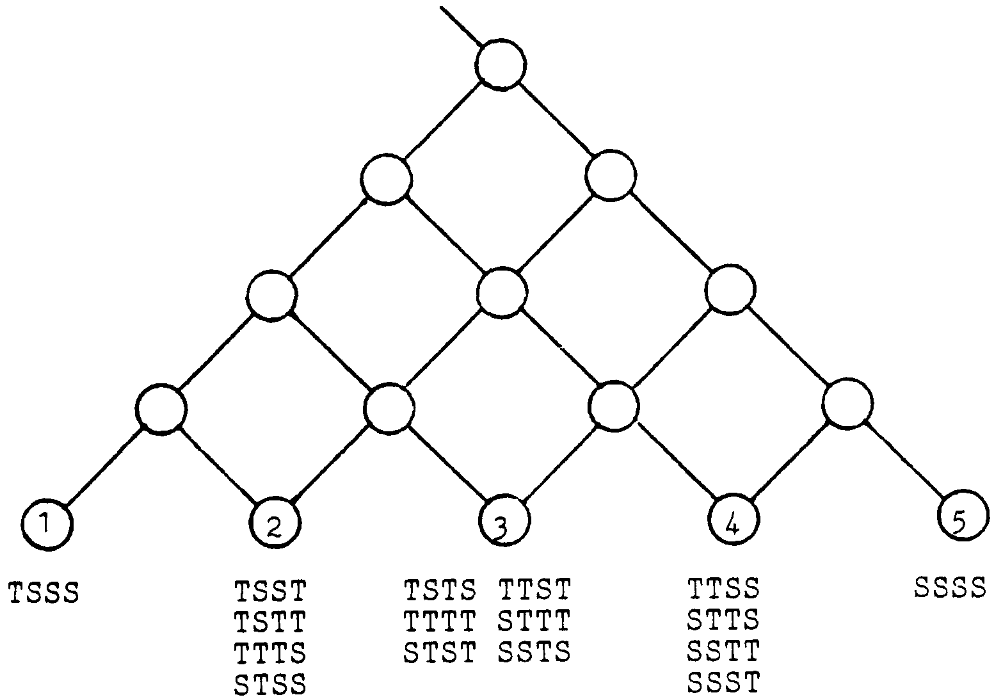

Figure 4.1 Achievable Evader Positions in Four-Step Game.

TABLE VIII

Markov-Hypothesis Strategy for Four-Step Game

$q_1 = 0.69681$ $q_3 = 0.70169$
$q_2 = 0.69681$ $q_4 = 0.69675$

$P(W=\hat{W}|STATE)$

$\hat{W}=$ STATE	1	2	3	4	5
SS	.10330	.18677	.23739	.23678	.23575
ST	.10329	.18511	.23709	.23710	.23740
TS	.10163	.18615	.23740	.23740	.23740
TT	.10331	.18512	.23709	.23710	.23738

TABLE IX

Three-Dependent Strategy to Four-Step Game

$q_1 = 0.69724$ $q_5 = 0.69728$
$q_2 = 0.69727$ $q_6 = 0.69727$
$q_3 = 0.70466$ $q_7 = 0.70469$
$q_5 = 0.69654$ $q_8 = 0.69724$

$P(W=\hat{W}|STATE)$

$\hat{W}=$ STATE	1	2	3	4	5
SSS	.10306	.18769	.23624	.23668	.23634
SST	.10294	.18508	.23733	.23731	.23733
STS	.10053	.18828	.23654	.23733	.23732
STT	.10329	.18518	.23731	.23712	.23709
TSS	.10457	.18826	.23622	.23612	.23482
TST	.10294	.18508	.23733	.23731	.23733
TTS	.10052	.18827	.23654	.23733	.23733
TTT	.10306	.18509	.23731	.23721	.23733

V. CONCLUSIONS AND REMARKS

The three-step pursuer-evader game remains unsolved. The investigation of the extended Markovian strategies has been shown to result in improved evader strategies over the Markov Hypothesis but is not known to provide a better strategy than the infinite memory strategy of Bram; in fact it is hypothesized that the n-dependent extended Markov strategy to the three-step game represents a finite approximation to the strategy of Bram. In this respect the results are not entirely disappointing in that they provide a finite strategy which appears to converge rather rapidly to a strategy equivalent to Bram's infinite memory strategy. The five-dependent strategy to the three-step game relies upon five distinct variables:

$$q_1 \qquad q_2 \qquad q_3 \qquad q_5 \qquad q_9$$

which provide an upper bound 0.28910 which is reasonably close to the bound of 0.28903 provided by Bram's infinite strategy. The near-optimal extended Markov strategies presented in Tables V, VI, and VIII represent local minima to the non-linear programming problem discussed in III.B. While these can be seen to represent improvements from the Markov Hypothesis strategy they may not be the globally minimum strategies within the extended Markov structure. As

the level of dependence in the extended Markov strategies increases the mathematical complexity increases disproportionately; only the apparent characteristic of these extended Markov strategies, discussed in III.E. makes them remotely attractive.

It still remains to be answered why the three-step game is apparently non-Markovian in its optimal evader strategy while the one and two-step games are Markovian. The evader strategy proposed by this thesis as well as the strategy described by Bouchoux represent abstractions from the strict Markov Hypothesis solution and although both strategies represent a lowering of the pursuer's maximum payoff, neither is as tight as the infinite strategy of Bram which is strictly non-Markovian in nature. While improved finite strategies may be possible by further abstraction from a strictly Markovian strategy, it has been conjectured that no finite strategy is optimal for the evader. This is known to be true for the pursuer since he must observe the evader for an ever-increasing length of time if he wishes to achieve optimality (with the exception of the one-step game where both sides have finite optimal strategies). Bouchoux suggests that a generalization of his sub-Markov strategy, involving three distinct Markov states each with some fixed probability of generating a straight or a turn, might provide a tighter bound on the game value due to its further abstraction from a Markov behavior. However, the mathematical

complexity of locating optimal or near-optimal strategies within this framework is considerable.

The four-step game appears even more difficult. The Markov Hypothesis solution is shown to be a sub-optimal strategy, being dominated by the three-dependent extended Markov strategy of Table IX. The strategies found to the four-step game in Tables VIII and IX appear to preclude an extension of Bram's infinite strategy to the four-step game. The apparent dissimilarity between the known near-optimal evader strategies from the two to three to four-step games is perplexing.

The discrete evasion game upon a two or three dimensional surface is another area which holds promise for future research. The work of Ferguson solves the two-step game for a special class of graphs he calls restricted n-graphs; however the two-step game upon more general two-dimensional surfaces, as well as the three-step game, are unsolved.

The discrete pursuer-evader game, as described by Isaacs in 1954, was generated as a simplification of a much more complex problem. The continuing mystery surrounding all but the simplest of these "simplified" games provides a wealth of opportunity and motivation for future research.

APPENDIX A

INVESTIGATION OF THE THREE-STEP EXTENDED MARKOV STRATEGY

In III.B., the general n-dependent extended Markov strategy was presented. The best solution found for the case n=3 is given in Table V. As stated earlier, this solution is not known to be optimal but can be shown to satisfy the first-order Kuhn-Tucker conditions (necessary but not sufficient) for a global minimum.

For the three-dependent case the problem may be stated as follows:

$$\min_{q_i} \left[\max_{\hat{W},\mathrm{STATE}} \{P(W=\hat{W}|\mathrm{STATE})\} \right]$$
$$\text{s.t.} \quad 0.0 \le q_i \le 1.0 \qquad i=1,2,3,4$$

There are sixteen separate functions (see Table IV), from which the maximum will be selected by the pursuer's choice of $\hat{W}$ and STATE (i.e. by his selection of aim point and time of fire), the evader must select the q_i's so as to minimize this maximum payoff. Let $f_1, f_2, \ldots, f_{16}$ represent the sixteen functions described in Table IV, then the problem becomes:

$$\min_{q_i} \left[\max_{\hat{W},\mathrm{STATE}} (f_1, f_2, \ldots, f_{16}) \right]$$
$$\text{s.t.} \quad 0.0 \le q_i \le 1.0 \qquad i=1,2,3,4$$

Introducing a dummy variable q_5, the above non-linear program may be equivalently written:

$$\min \quad q_5$$

$$\begin{array}{llll} \text{s.t.} & f_j - q_5 \le 0.0 & & j=1\text{-}16 \\ & q_i - 1.0 \le 0.0 & & i=1\text{-}4 \\ & q_i \ge 0.0 & & 1=1\text{-}4 \end{array}$$

The structure of this problem allows some additional conditions to be placed upon the optimal solution;

$$0.0 < q_i < 1.0 \qquad i=1,2,3,4.$$

Close inspection of the functions, f_j, show that if:

$$q_i = 0.0 \qquad \text{or}$$

$$p_i = 1.0-q_i = 0.0$$

then at least one of the f_j's will have a value of 0.0. If any f_j=0.0 then the remaining three f_j's associated with the same initial state must sum to 1.0, since for any initial state:

$$P(W=1,2,3 \text{ or } 4|STATE) = 1.0$$

The minimum of the maximum of three non-negative numbers which sum to 1.0 must be at least 1/3, which is greater than the known upper bound on the value of the game. Therefore:

$$0.0 < q_i < 1.0 \qquad i=1\text{-}4$$

Based upon the above characteristic of the problem the constraints;

$$q_i - 1.0 \leq 0.0 \qquad i=1-4$$

will not be binding at the optimal solution and may be dropped without consequence, resulting in:

$$\begin{array}{lll} \min & q_5 & \\ \text{s.t.} & f_j - q_5 \leq 0.0 & j=1-16 \\ & q_i \geq 0.0 & i=1-5 \end{array}$$

The first-order Kuhn-Tucker conditions for the above problem require that, at an optimal point, there exist a set of λ's such that:

$$\frac{\partial L}{\partial q_i} \geq 0.0 \qquad\qquad \frac{\partial L}{\partial \lambda_j} \geq 0.0$$

$$q_i \frac{\partial L}{\partial q_i} = 0.0 \qquad\qquad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0.0$$

$$q_i \geq 0.0 \qquad\qquad \lambda_j \leq 0.0$$

$$i = 1-5 \qquad\qquad j = 1-16$$

where:

$$L(\bar{q},\bar{\lambda}) = q_5 - \sum_{j=1}^{16} \lambda_j (f_j - q_5) \ .$$

These conditions may be further modified:

$$\frac{\partial L}{\partial q_i} = 0.0$$

$$q_i \frac{\partial L}{\partial q_i} = 0.0$$

$$q_i > 0.0$$

$$i = 1\text{-}5$$

In the proposed near-optimal solution in Table V, seven of the sixteen inequality constraints are binding; that is:

$$f_4 = f_6 = f_7 = f_8 = f_{14} = f_{15} = f_{16} = q_5 = 0.28964$$

the remaining nine constraints are slack, it follows that:

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = \lambda_{13} = 0.0$$

The proposed solution must therefore satisfy the following conditions:

$$\frac{\partial L}{\partial q_i} = 0.0 \qquad i = 1,5 \qquad \lambda_j \leq 0.0 \qquad j = 4,6,7,8,14,15,16$$

with the substitution of the values,

$$q_1 = 0.66163 \quad q_2 = 0.70054 \quad q_3 = 0.62489 \quad q_4 = 0.70054$$

the five constraints $(\frac{\partial L}{\partial q_i} = 0.0)$, *become a set of five* linear equations in seven unknowns (λ_4, λ_6, λ_7, λ_8, λ_{14}, λ_{15}, λ_{16}). Any solution to this set of equations which also satisfies the condition:

$$\lambda_j \leq 0.0 \qquad j=4,6,7,8,14,15,16$$

will satisfy the modified Kuhn-Tucker conditions. Using linear programing methods, such a set of λ's was found, thereby verifying the satisfaction of the Kuhn-Tucker conditions at the proposed three-dependent strategy of Table V. The near-optimal solutions to the four and five-dependent strategies (Tables VI and VII) could be analyzed in a similar manner.

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